Scaling, self-similar solutions and shock waves for V-shaped field potentials

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We investigate a $(1+1)$-dimensional nonlinear field theoretic model with the field potential $V(\phi)=|\phi|$. It can be obtained as the universal small amplitude limit in a class of models with potentials which are symmetrically V-shaped at their minima, or as a continuum limit of certain mechanical system with an infinite number of degrees of freedom. We point out that the model has an interesting scaling symmetry. One consequence of that is the existence of self-similar solutions of the corresponding field equations. We also find discontinuous solutions of shock wave type.

I. INTRODUCTION

Classical $(1+1)$-dimensional field theoretic models with nonlinear field equations appear in many problems in physics. Well-known examples are provided by sine-Gordon and ϕ^4 models, which have plenty of applications: from Josephson junctions to macroscopic mechanical systems, see, e.g., [1,2]. Very rich contents of these models have been explored in great detail. In the ultralocal limit, i.e., when gradient terms in the field energies are neglected, both models can be regarded as describing infinite sets of simple uncoupled one-dimensional mechanical systems: pendulums or anharmonic oscillators, respectively.[1] In the present paper we consider a model which in that limit describes an infinite set of coupled balls bouncing vertically from a floor, but of course this is not the reason for investigating it in detail. The actual motivation is that the model possesses very interesting scaling symmetry. Furthermore, it universally describes the small amplitude sector in models with nonsmooth field potentials. In a sense, it is a counterpart of free field models which describe the small amplitude sectors in the case of smooth field potentials.

Specifically, we consider the $(1+1)$-dimensional, classical field-theoretic model with the field potential

$$V(\phi) = |\phi|, \qquad (1)$$

and the corresponding field equation

$$\frac{\partial^2 \phi(\xi,\tau)}{\partial \tau^2} - \frac{\partial^2 \phi(\xi,\tau)}{\partial \xi^2} = -\operatorname{sgn}[\phi(\xi,\tau)], \qquad (2)$$

where ϕ is a real scalar field. The sgn function has the values ± 1 when $\phi \neq 0$, and 0 if $\phi=0$. The potential V reaches its absolute minimum at $\phi=0$, where it is not differentiable—the left and right derivatives of V have different values

$$\left. \frac{dV(\phi)}{d\phi} \right|_{\phi=0\pm} = \pm 1.$$

Hence, the potential is symetrically V shaped at the minimum. This feature distinguishes our model from the majority of models of condensed matter physics and particle physics. Notice also that Eq. (2) cannot be linearized even if ϕ is arbitrarily small.

The nonsmooth field potential (1) appears in an effective field theoretic description (in a continuum approximation) of a discrete mechanical system composed of balls, which are connected by elastic strings and bounce vertically from the floor, see Fig. 1. Another system with a V-shaped potential—elastically coupled pendulums bouncing between two rods—is described in [3]. Moreover, V-shaped field potentials appear in theoretical investigations of pinning phenomenon of extended objects, see, e.g., [4]. These facts prove that even though the model might look rather exotic, it is not unrelated to the physical world.

The model is quite intriguing. First, it is distinguished by the above-mentioned universality. Second, apart from the obvious $(1+1)$-dimensional Poincarè invariance, it possesses the exact scaling symmetry of the "on shell" type, which resembles a little bit the approximate scaling symmetry of Navier-Stokes equations for a fluid in the turbulent regime.
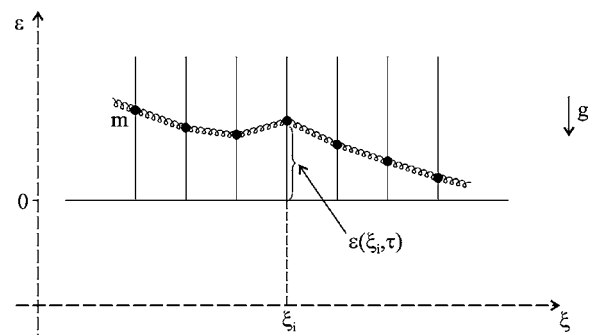

FIG. 1. The system of balls connected by strings. The balls can move along the vertical poles (the continuous vertical lines) without friction. They elastically bounce from the floor which is depicted as the continuous horizontal line. Here $\epsilon \sim h$, $\xi \sim x$, and $\tau \sim t$ are the dimensionless variables introduced in the text.

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[1]Let us notice that the ultralocal limit yields systems with an infinite number of degrees of freedom. It should not be confused with the infinite wavelength limit which leads to systems with one degree of freedom: a single pendulum or anharmonic oscillator.

This symmetry is the most profound feature of the model. Because of its presence we may expect that there exist so called self-similar solutions of the field equation, see, e.g., [5] for the relevant mathematical concepts. Indeed, we have found such solutions. They have quite interesting structures both from the physical and mathematical viewpoints. Another very interesting feature of the field Eq. (2) is that it admits shock waves: finite discontinuous steps of the field ϕ which move with the velocities ± 1 ("the velocities of light") and leave wakes behind them. One more reason for our interest in that model is that, to the best of our knowledge, effects of the nonlinearity of the $\mathrm{sgn}(\phi)$ type in field equations have not been thoroughly investigated yet.

The plan of our paper is as follows. In Sec. II, two derivations of the model with potential (1) are given: (a) the continuum approximation to the system of bouncing balls, (b) the small amplitude approximation to a general symmetrically V-shaped potential. Section III is devoted to the scaling invariance and to the self-similar solutions. The shock waves are discussed in Sec. IV. Section V contains a summary and certain remarks.

II. DERIVATION OF THE MODEL

A. Continuum limit in the system of elastically coupled, vertically bouncing balls

Balls of mass m can move along vertical poles which are fastened to the floor at the points $x_i = ai$, $i = 0, \pm 1, \pm 2, \ldots$, lying along a straight line. The balls bounce elastically from the floor. Each of them is connected with its nearest neighbors by straight linear segments of elastic strings. The system has one degree of freedom per ball. It is represented by the elevation $h(x_i, t)$ above the floor of the ball at the point x_i and the time t, see Fig. 1.

Let us forget for the moment about the floor. Then each ball is subject to the force of gravity (represented by the gravitational acceleration g), and to the elastic forces from the two attached strings. It is an elementary exercise to write the Newton equations of motion for the balls. Next, we pass to the dimensionless variables

$$\tau = \sqrt{\frac{g}{h_0}} t, \quad \xi = \sqrt{\frac{mg}{\kappa a^2 h_0}} x, \quad \epsilon(\xi, \tau) = h_0^{-1} h(x_i, t),$$

where κ is the elasticity constant for the strings, h_0 is a unit of length along the poles. Finally, we take the continuum limit

$$a \to 0, \quad \frac{m}{\kappa a^2} = \text{const.}$$

As the result, we obtain the equations of motion for the system of balls in the form

$$\frac{\partial^2 \epsilon(\xi, \tau)}{\partial \tau^2} - \frac{\partial^2 \epsilon(\xi, \tau)}{\partial \xi^2} = -1, \tag{3}$$

provided that $\epsilon(\xi, \tau) > 0$. This last condition is due to the fact that we have not taken into account the presence of the floor.

In case $\epsilon = 0$ at certain point ξ_0 and time τ_0, Eq. (2) is replaced at that point by the elastic bouncing condition

$$\frac{\partial \epsilon}{\partial \tau}(\xi_0, \tau_0) \to -\frac{\partial \epsilon}{\partial \tau}(\xi_0, \tau_0) \quad \text{when } \epsilon(\xi_0, \tau_0) = 0. \tag{4}$$

We can remove this cumbersome condition with the help of a trick used earlier for the system of pendulums [3,6] and called "the unfolding" of the model. In the unfolded model instead of the field $\epsilon(\xi, \tau) \geqslant 0$ we have a new field $\underline{\epsilon}(\xi, \tau)$ which can take arbitrary real values. By assumption, the evolution equation for $\underline{\epsilon}$ has the form (2)

$$\frac{\partial^2 \underline{\epsilon}(\xi, \tau)}{\partial \tau^2} - \frac{\partial^2 \underline{\epsilon}(\xi, \tau)}{\partial \xi^2} = -\mathrm{sgn}[\underline{\epsilon}(\xi, \tau)]. \tag{5}$$

The field potential corresponding to the right-hand side (r.h.s.) of Eq. (5) has the perfect symmetrically V-shaped form, namely,

$$\underline{V}(\underline{\epsilon}) = |\underline{\epsilon}|. \tag{6}$$

Because the r.h.s. of Eq. (5) is finite, the velocity $\partial \underline{\epsilon} / \partial \tau$ is a continuous function of time τ. On the other hand, the two second order derivatives of $\underline{\epsilon}$ cannot be both continuous when $\underline{\epsilon} = 0$ because the r.h.s. of Eq. (5) is discontinuous.

The original field ϵ is related to $\underline{\epsilon}$ by the following formula (the folding transformation):

$$\epsilon(\xi, \tau) = |\underline{\epsilon}(\xi, \tau)|. \tag{7}$$

It is clear that Eq. (5) and formula (7) imply Eq. (3) for ϵ when $\epsilon \neq 0$. The elastic bouncing condition (4) follows from formula (7) and from the continuity of $\partial \underline{\epsilon} / \partial \tau$

$$\partial \epsilon / \partial \tau = \mathrm{sgn}(\underline{\epsilon}) \, \partial \underline{\epsilon} / \partial \tau,$$

hence the sgn of $\partial \epsilon / \partial \tau$ changes when $\underline{\epsilon}$ (and ϵ) passes through 0.

The derivation of Eq. (5) presented above applies also to the phenomenon of pinning of an elastic string (e.g., a vortex) by a rectilinear impurity. In that case the gravitational force is replaced by the pinning force, and the system of balls and springs by the string.

B. Small amplitude sector for the symmetrically V-shaped potential

Let us consider a general field equation of the form

$$\frac{\partial^2 \psi(\xi, \tau)}{\partial \tau^2} - \frac{\partial^2 \psi(\xi, \tau)}{\partial \xi^2} + V'(\psi) = 0, \tag{8}$$

where the field potential $V(\psi)$ is V shaped at the minimum located at $\psi = 0$, see Fig. 2.

We are interested in the small amplitude oscillations of the field ψ around the minimum of the field potential V. Because $\psi \approx 0$, we may replace V' by its piecewise linear approximate form

$$V(\psi) \cong V'(0+)\psi\Theta(\psi) - |V'(0-)|\psi\Theta(-\psi),$$

where Θ denotes the step function. Then, Eq. (8) is replaced by
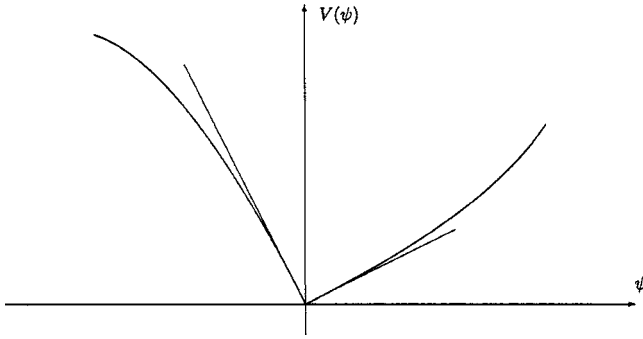
FIG. 2. Generic V-shaped potential and the piecewise linear approximation.

$$\frac{\partial^2 \psi}{\partial \tau^2} - \frac{\partial^2 \psi}{\partial \xi^2} = -V'(0+)\Theta(\psi) + |V'(0-)|\Theta(-\psi). \quad (9)$$

Notice that this equation remains nonlinear for arbitrarily small ψ—there is no linear regime for the small oscillations. After rescaling the field ψ

$$\psi(\xi, \tau) = V'(0+)\eta(\xi, \tau),$$

we obtain the following equation:

$$\frac{\partial^2 \eta}{\partial \tau^2} - \frac{\partial^2 \eta}{\partial \xi^2} = -\text{sgn}(\eta) + \left(\frac{|V'(0-)|}{V'(0+)} - 1\right)\Theta(-\eta). \quad (10)$$

Here we have used the formula $\Theta[V'(0+)\eta] = \Theta(\eta)$ which is valid because $V'(0+) > 0$. In the particular case of symmetrically V-shaped potential $|V'(0-)| = V'(0+)$ and we obtain Eq. (2).

Example of physical system for which Eq. (2) can be applied only in the small amplitude limit can be found in [3]. It is composed of elastically coupled pendulums which bounce from a rod.

III. SCALE INVARIANCE AND THE SELF-SIMILAR SOLUTIONS

Equation (2) has the scaling symmetry: If $\phi(\xi, \tau)$ is a solution of it, then

$$\phi_\lambda(\xi, \tau) \stackrel{df}{=} \lambda^2 \phi\left(\frac{\xi}{\lambda}, \frac{\tau}{\lambda}\right), \quad (11)$$

where $\lambda > 0$ is an arbitrary positive number, obeys Eq. (2) too.[2] The field energy

$$E[\phi] = \frac{1}{2}\int d\xi[(\partial_\tau \phi)^2 + (\partial_\xi \phi)^2] + \int d\xi V(\phi)$$

scales as follows:

$$E[\phi_\lambda] = \lambda^3 E[\phi].$$

When $\lambda \to 0$ the solutions $\epsilon_\lambda(\xi, \tau)$ are in general characterized by high frequencies, short wavelengths, and small

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[2]Also Eq. (10) has such symmetry.

energies. This is in contrast to, e.g., the massless ϕ^4 model which also has a scaling symmetry, but in that case the scaling points to low frequencies and large wavelengths when the energy $E[\phi_\lambda] \to 0$. Let us also remark that this reminds the phenomenon of turbulence—even more so when we recall that the Navier-Stokes equations in the high Reynolds number regime have a scale invariance similar to the one described above, see, e.g., the Introduction section in [7].

The immediate consequence of the scale invariance is the lack of a characteristic frequency or energy scale. For example, one can find infinite periodic running wave solutions to Eq. (2) which have the dispersion relation of the form

$$\omega^2 - k^2 = \mu^2,$$

where μ^2 can take any value from the interval $(0, \infty)$ [3].

Let us stress that the symmetry transformation (11) is defined only in the space of solutions of the field Eq. (2). The action functional corresponding to that equation

$$S[\phi] = \int d\tau d\xi\left[\frac{1}{2}(\partial_\tau \phi)^2 - \frac{1}{2}(\partial_\xi \phi)^2 - |\phi|\right],$$

which can be calculated also for fields which do not obey Eq. (2), is not invariant with respect to the transformations (11). Therefore, the symmetry (11) is not of the Noether type. In such a case one often says that the symmetry is of the "on shell" kind, see, e.g., [8].

As always when there is a symmetry, one can find solutions of the pertinent field equation which are invariant under the symmetry transformation. In order to find the self-similar solutions of the field Eq. (2), that is, the ones which are invariant under the scale transformations (11), we adopt the following scale invariant Ansatz for the solution

$$\phi(\xi, \tau) = \xi^2 S(y), \quad y = \frac{\tau}{\xi}.$$

In consequence, Eq. (2) is reduced to the ordinary differential equation for the function $S(y)$

$$(1 - y^2)S'' + 2yS' - 2S = -\text{sgn}(S). \quad (12)$$

This equation has the following polynomial solutions:

$$\text{when } S > 0: \quad S(y) = -\frac{1}{2}\beta(y^2 + 1) + \frac{\alpha}{2}y + \frac{1}{2},$$

$$\text{when } S < 0: \quad S(y) = \frac{1}{2}\beta'(y^2 + 1) - \frac{\alpha'}{2}y - \frac{1}{2},$$

where $\alpha, \beta, \alpha', \beta'$ are arbitrary constants. It has also the trivial solution

$$S_0(y) = 0.$$

The polynomial solutions are valid on appropriate finite intervals of the y axis which are determined by the conditions $S > 0$ or $S < 0$, respectively.

Let us try to put together the polynomial solutions in order to cover as large as possible an interval of the y axis. The resulting solution has the form depicted in Fig. 3, where S_k, $k = 1, 2, \ldots$, denote the polynomial solutions. We assume that
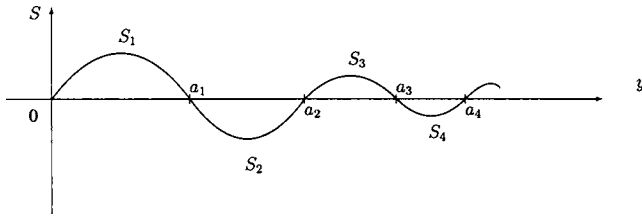
FIG. 3. The piecewise polynomial solution. a_k are the matching points for the consecutive polynomials S_k.

$S(0)=0$ because $y=0$ corresponds to the spatial infinity, $\xi=\pm\infty$, and we would like to minimize the growth of $\phi(\xi,\tau)$ in that region.

Let us write the polynomial solutions in the form

$$S_k(y) = \frac{1}{2}(-1)^k[\beta_k(y^2+1) - \alpha_k y - 1].$$

Equation (12) implies that the following matching conditions at the points $y=a_k$ have to be satisfied:

$$S_k(a_k) = 0 = S_{k+1}(a_k), \quad S'_k(a_k) = S'_{k+1}(a_k), \quad (13)$$

provided that $a_k \neq \pm 1$. The points $y=\pm 1$ are exceptional because then the coefficient in front of the second derivative term in Eq. (12) vanishes. This has the consequence that the first derivative dS/dy of the solution does not have to be continuous at the points $y=\pm 1$. Below we shall make use of this possibility.

The matching conditions (13) give the recurrence relations:

$$\alpha_{k+1} = \frac{4a_k}{1-a_k^2} - \alpha_k, \quad \beta_{k+1} = \frac{2}{1-a_k^2} - \beta_k,$$

$$a_{k+1} = \frac{2a_k - (1+a_k^2)a_{k-1}}{1+a_k^2 - 2a_k a_{k-1}},$$

where $\alpha_1 = a_1 > 0$, $\beta_1 = 1$, $a_0 = 0$. We also assume that $a_1 < 1$ in order to ensure that $a_2 > a_1$. Rather unexpectedly, we have found exact solutions of these relations—it has turned out that a glance at explicit forms of several initial α_k, β_k, and a_k is sufficient in order to guess the solutions. They have the following form:

$$\alpha_k = \frac{1}{2}(1+a_1)q^{1-k}(1-q^{2k-1}),$$

$$\beta_k = \frac{1}{4}(1-a_1)(1+q^{k-1})(1+q^{-k}), \quad (14)$$
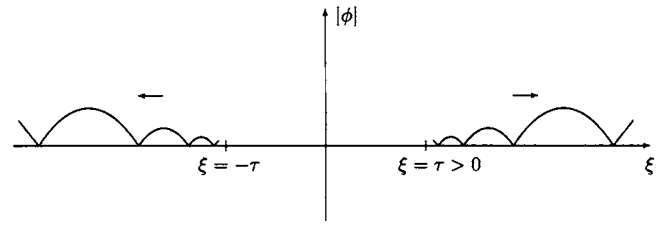
$$a_k = \frac{1-q^k}{1+q^k}, \quad (15)$$

where

FIG. 4. The self-similar solution. The two arrows indicate that the waves emanate from the points which move along the ξ axis: $\xi = \pm\tau$. We plot $|\phi|$ because this is the physical variable in the case of system of balls—see the relation (7).

$$q = \frac{1-a_1}{1+a_1}.$$

Of course, this solution can be checked by substituting it into the recurrence relations.

Because $0 < q < 1$, formula (15) implies that $a_k < 1$, and that $a_k \rightarrow 1$ when $k \rightarrow \infty$. Therefore, the piecewise polynomial solution constructed above covers only the interval $[0,1)$. For this reason we introduce a special notation for it, namely, $S_{pp}(y)$. Simple calculations show that S_{pp} vanishes when $y \rightarrow 1$. On the other hand, the first derivative $S'_{pp}(y)$ does not vanish in that limit, but it remains finite.

This solution can be extended to the full range of y. First, we extend it to the interval $(-1,1)$ by taking as the solution in the interval $(-1, 0]$ the function $-S_{pp}(-y)$, which smoothly matches the function $S_{pp}(y)$ at the point $y=0$. Next, we take the trivial solution $S_0(y)=0$ for $y \leq -1$ and $y \geq 1$. This is possible because $S_{pp}(y)$ and $-S_{pp}(-y)$ vanish in the limit $y \rightarrow 1$. The first derivatives of the functions $S_{pp}(y)$ and $S_0(y)$ at the points $y=\pm 1$ do not have to be equal because Eq. (12) allows for a finite jump of the first derivative $S'(y)$ when $y^2=1$.

The parameter a_1 remains free except for the restriction $0 < a_1 < 1$, so we actually obtain a family of self-similar solutions. A snapshot of the solution at certain $\tau > 0$ is depicted in Fig. 4. It is composed of infinitely many quadratic polynomials in ξ taken on domains which become smaller and smaller when $\xi \rightarrow \tau+$ or $\xi \rightarrow -\tau-$. For $\xi \rightarrow \pm\infty$ $|\phi|$ approaches the function $a_1\tau|\xi|/2 - \tau^2/2$. The zeros of ϕ lie at $\xi_k(\tau) = \tau/a_k$. Hence, $\dot{\xi}_k = 1/a_k > 1$ – the zeros move with "superluminal" velocities. The function $\phi(\xi,\tau)$ exactly vanishes for $|\xi| < \tau$, i.e., inside the "light cone." This follows from the fact that the restriction $-1 < y < 1$ is equivalent to $|\xi| > \tau$.

The solution presented in Fig. 4 is symmetric with respect to $\xi \rightarrow -\xi$. It turns out that the left- and right-hand halves of this solution taken separately are solutions, too: $\phi(\xi,\tau) = \Theta(\xi-\tau)\xi^2 S(\frac{\tau}{\xi})$ and $\phi(\xi,\tau) = \Theta(\xi+\tau)\xi^2 S(\frac{\tau}{\xi})$ also obey Eq. (2).

The full self-similar solution described above cannot be physically realized in a physical system. The reason is that it has infinite energy—this is the general feature of nontrivial self-similar solutions. Nevertheless, finite pieces of it can be observed if the dynamics of the model is local, as in our case. Namely, if a physical finite energy initial configuration differs from our solution only at the very large values of $|\xi|$

($|\xi| \gg \tau$), this difference will be seen in the region of finite $|\xi|$ ($|\xi| \sim \tau$) only after a certain finite time interval. During that time interval the piece of the self-similar solution describes the evolution of the finite $|\xi|$ part of the system very well. Information about uses of self-similar solutions can be found, e.g., in [5].

IV. SYMMETRIC SHOCK WAVES

Another interesting feature of the system governed by Eq. (2) is that it can support shock waves. Such waves are not invariant under the scale transformation (11).

The Ansatz

$$\phi(\xi,\tau) = \Theta(-z)W(z), \quad z = \frac{1}{4}(\xi^2 - \tau^2), \quad (16)$$

reduces Eq. (2) to the following ordinary differential equation:

$$zW'' + W' = \text{sgn}(W), \quad (17)$$

where $'$ denotes the derivative d/dz. We have used the following formulas:

$$\Theta'(-z) = -\delta(z), \quad z\delta(z) = 0, \quad z\delta'(z) = -\delta(z),$$

$$\text{sgn}[\Theta(-z)W(z)] = \Theta(-z)\text{sgn}[W(z)].$$

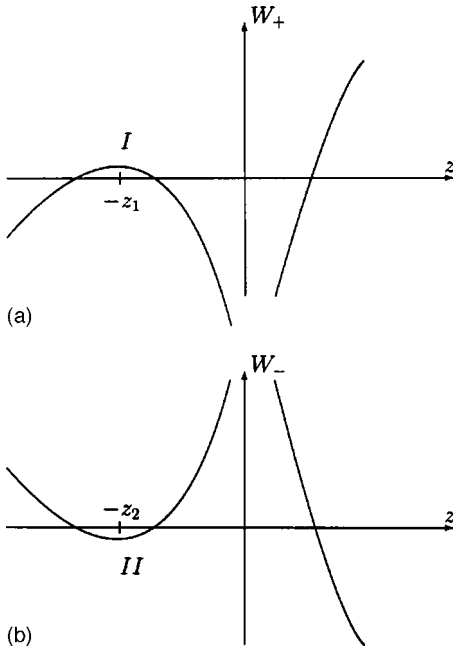Equation (17) has the following partial solutions:

FIG. 5. The partial solutions $W_\pm$. The shock wave is obtained by smooth matching of infinitely many copies of the parts *I*, *II*.
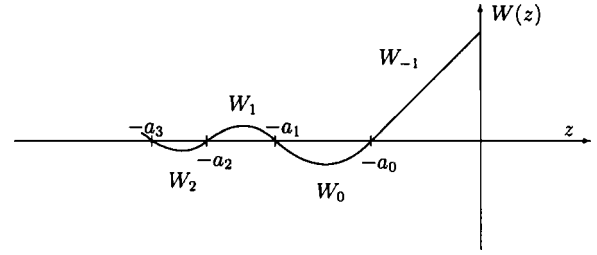

FIG. 6. The scheme for matching the parts *I*, *II* of the partial solutions $W_\pm$.

$$\text{when } W > 0: \quad W_+(z) = z + z_1 + z_1 \ln\left|\frac{z}{z_1}\right| + d_1,$$

$$\text{when } W < 0: \quad W_-(z) = -z - z_2 - z_2 \ln\left|\frac{z}{z_2}\right| - d_2,$$

where d_1, d_2, z_1, z_2 are constants. These solutions are defined on finite intervals of the z axis, see Fig. 5.

Putting the solutions $W_\pm$ together we obtain the solution which is defined for all $z \leq 0$. Because of the step function in the Ansatz we do not need to know $W(z)$ for $z > 0$. In order to analyze the matching conditions for the partial solutions we introduce the following notation:

$$W_{-1}(z) = z + a_0, \quad W_k(z) = (-1)^k\left(z + a_k + z_k \ln\frac{|z|}{a_k}\right),$$

where $a_k > 0$. The matching conditions at the points $-a_k$, $k = 0, 1, \ldots$, have the form

$$\text{(a)} \quad W_k(-a_k) = 0 = W_{k-1}(-a_k),$$

$$\text{(b)} \quad W_k'(-a_k) = W_{k-1}'(-a_k),$$

see Fig. 6.

The matching conditions (b) for the derivatives yield the following recurrence relation:

$$z_{k+1} = 2a_{k+1} - z_k, \quad \text{where } z_0 = 2a_0. \quad (18)$$

Here $a_0 > 0$ is a free parameter. The matching conditions (a) for the functions W_k also give a recurrence relation, but it is
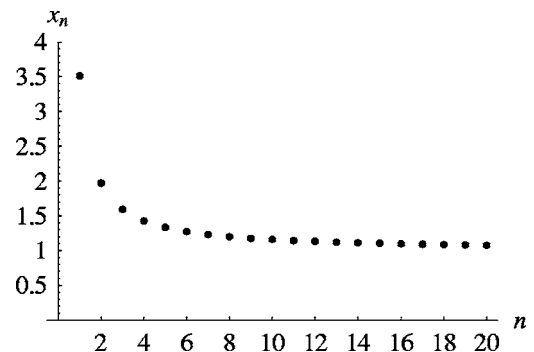

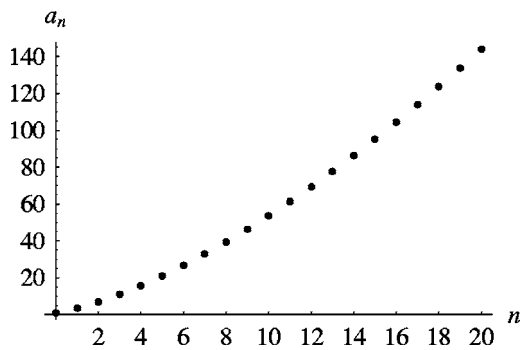FIG. 7. The numerical solutions x_n of Eqs. (20).

FIG. 8. The numerical results for the coefficients a_n.

much more complicated. In order to write it in a transparent form, instead of a_k we equivalently use x_k defined by the following formula:

$$a_{k+1} = a_0 x_1 x_2 \cdots x_{k+1}, \quad x_k > 0, \quad k = 0, 1, \ldots, \quad (19)$$

Then, conditions (a) are equivalent to the following infinite set of equations for $x_1, x_2, \ldots,$:

$$x_1 = 1 + 2 \ln x_1,$$

$$x_2 = 1 + 2\left(1 - \frac{1}{x_1}\right) \ln x_2,$$

$$\vdots$$

$$x_k = 1 + 2\left(1 - \frac{1}{x_{k-1}} + \frac{1}{x_{k-1}x_{k-2}}\right.$$
$$\left. - \cdots \pm \frac{1}{x_{k-1}x_{k-2} \cdot \ldots \cdot x_2 x_1}\right) \ln x_k.$$

$$\vdots \quad (20)$$

These equations can be rewritten in terms of Lambert W function [9], but we will not give here the related formulas. Numerical solutions of the first 20 Eqs. (20) are presented in Fig. 7, while Fig. 8 shows the corresponding values of a_n.

The numerical calculations convincingly indicate that $x_n \to 1$ as n increases. Unfortunately, this is not sufficient to determine whether a_n, related to x_k by formula (19), increase to infinity or converge to a finite value. We have numerically investigated the sums $\sum_{n=1}^{N} \ln x_n$ up to $N=250$. The results suggest that the sum becomes divergent when $N \to \infty$. Therefore, we conjecture that a_n grow to infinity when $n \to \infty$. In

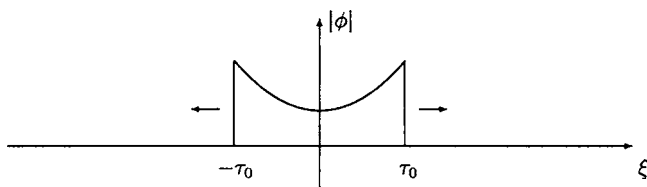
FIG. 9. The symmetric shock wave at an initial time τ_0 such that $0 < \tau_0 < 2\sqrt{a_0}$. The arrows indicate the directions in which the wave fronts move. $\phi(\xi, \tau_0) = 0$ for $|\xi| > \tau_0$.
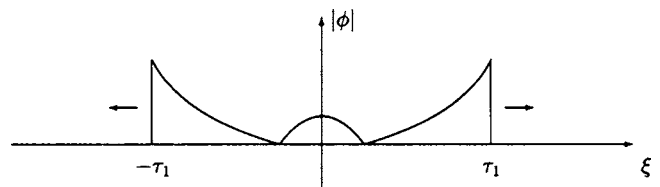

FIG. 10. The symmetric shock wave at a time $\tau_1 > \tau_0$ such that $2\sqrt{a_1} > \tau_1 > 2\sqrt{a_0}$. Now $\phi(\xi, \tau_1) = 0$ for $|\xi| > \tau_1$.

this case the function $W(z)$ is defined in the whole half-axis $z \leq 0$, and formula (16) gives the solution for all values of z.

The snapshots of the shock wave at three times are shown in Figs. 9–11.

The features of the shock wave shown on Figs. 9–11 can be obtained as follows. The condition $z \leq 0$ implies that $\phi(\xi, \tau)$ vanishes outside the "light cone," i.e., when $\xi^2 > \tau^2$. The "light cone" corresponds to $z = 0$, where $\phi = a_0$. The constant $a_0 > 0$ is a free parameter. It gives the height of jump of the value of the field ϕ on the "light cone". Thus, $\Theta(-z)W(z)$ describes a shock wave, symmetric with respect to $\xi \to -\xi$ and restricted to the "light-cone". The velocities of the steps (shock fronts) are equal to ± 1.

At the given time $\tau > 0$ the physically visible range of z is finite—it is given by the inequalities $-\tau^2/4 \leq z \leq 0$. Therefore, we can see only a finite number of zeros of ϕ, but that number grows indefinitely with time. The positions of the zeros of the field ϕ are determined from the equations $z = -a_k$. Hence, they are given by the formulas

$$\xi_k(\tau) = \pm \sqrt{\tau^2 - 4a_k}.$$

Their absolute velocities are greater than 1. Thus, the zeros move behind the front and faster than it, but they never catch up with the front. The first zero of ϕ appears at the point $\xi = 0$ at the time $\tau_i = 2\sqrt{a_0}$. Notice that it is by factor $\sqrt{2}$ larger than the time $\sqrt{2a_0}$ needed by a freely falling ball in order to hit the floor if its initial velocity is equal to zero and the initial elevation above the floor to a_0. The reason for this difference is obvious: The ball located at $\xi = 0$, which indeed has the elevation a_0 and the vanishing velocity in the limit $\tau \to 0$, does not fall freely because it is coupled to its neighbours.

The discontinuous shock waves have infinite energy because the gradient energy density at the shock fronts is infinite. Nevertheless, similar as the self-similar solutions, they can be very helpful if used with due care.
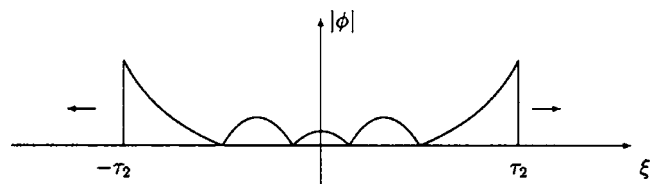

FIG. 11. The symmetric shock wave at a time $\tau_2 > \tau_1$. At later times more zeros of ϕ appear.

V. SUMMARY AND REMARKS

The nonlinear term $-\text{sgn}(\phi)$ in the evolution Eq. (2) rarely is considered in field theory. Nevertheless, it is quite interesting, in particular because it has the scaling symmetry (11), and in consequence allows for the self-similar waves. Furthermore, we have shown in an earlier paper, see [3], that in other models with nonlinearity of that kind there exist static kinks which have strictly finite extension (no exponential tails)—for that reason they are called compactons.

Systems with nonsmooth potentials are quite common in classical mechanics. Especially interesting are so called impacting systems, for example, a harmonically oscillating particle which can hit a wall when it moves sufficiently far from its equilibrium position. Recently, such systems have been investigated in connection with chaotic behavior and so called grazing bifurcation, see, e.g., [10,11]. In the present paper we have considered the field-theoretic counterpart of such systems.

There are many interesting possibilities for future investigations of the models with V-shaped field potentials. Let us point two of them:

(1) It is not known whether the models of that kind can be integrable.

(2) What are properties of quantum version of the (1+1)-dimensional model with the potential $V(\phi)=|\phi|$? We expect that the scaling symmetry (11) is broken in the quantum theory. What is the resulting mass scale?

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